Integrating SMT with Theorem Proving for AMS Verification

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Outline

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✿ **Contributions**

- Integrating SMT with Theorem Proving, challenges and solutions
- Verifying global convergence of a Digital Phase-Locked Loop(DPLL) using recurrence
- Conclusion

Contributions

- Combine industrial strength SMT solver with industrial strength theorem prover.
- Model state-of-the-art DPLL with recurrences.
- Proof of global convergence.
- Able to prove design with parameter variation.

Outline

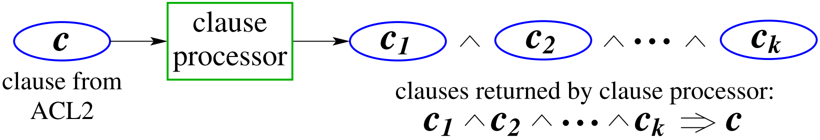
Integrating SMT with Theorem Proving for AMS Verification

- Contributions
- **Integrating SMT with Theorem Proving**
  - **Why combine Z3 and ACL2?**
  - **Software framework and technical challenges**
- Verifying global convergence of a Digital Phase-Locked Loop(DPLL) using recurrence
- Conclusion

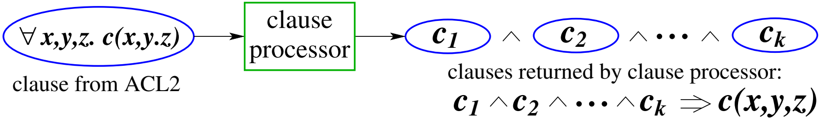
- *Satisfiability Modulo Theories (SMT)* problem is a unified decision procedure for logical formulas which combines solvers for a rich set of background theories.
- Possible theories: propositional logic, arithmetic, uninterpreted functions, bitvectors theories etc.
- *Z3, Microsoft Research* [MB08, JM12]. Non-linear arithmetic theories, suitable for AMS design with non-linear dynamics.
- Lack of:
  - Induction proof
  - Structured proof

- *Theorem proving* is a technique for proving a set of theorems by building upon a set of basic axioms and use of logic rules, e.g. rewrite rules, induction.
- In order to prove a final theorem, one looks at what is needed and develops a set of lemmas.
- ACL2, University of Texas at Austin.[KM97]
- But working through complicated boolean formulas, systems of inequalites, etc., can be extremely tedious.
- ACL2 and Z3 complement each other:
  - ACL2 provides structured proofs and induction proofs.
  - Z3 discharges complicated/tedious systems of inequalities.

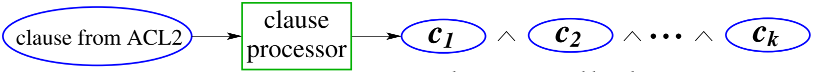
SMT and Theorem Proving - clause processor



- A clause processor takes the goal one wants to prove and decomposes the goal into a conjunction of subgoals. Each subgoal is a called a clause.
- ACL2 supports two kinds of clause processors:
  - A **verified** clause processor is written in Lisp and proven correct within ACL2.
  - A **trusted** clause processor is anything else.
    Theorems whose proofs rely on a trusted clause processor are tagged accordingly.
- We integrate Z3 into ACL2 as a trusted clause processor.



- Challenge: ACL2 has rationals and Z3 has reals.
  - In ACL2, $\neg \exists x.\ x^2 = 2$ is a theorem.
  - In Z3, $\exists x.\ x^2 = 2$ is a theorem.
- Solution: only use Z3 to prove propositions where all variables are universally quantified.



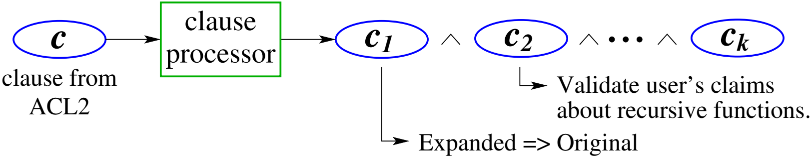

```
(implies (and (rationalp x)
              (rationalp y)
              (rationalp z))
         (c x y z))
```

clauses returned by clause processor:

$$c_1 \wedge c_2 \wedge \cdots \wedge c_k \Rightarrow \mathbf{c(x,y,z)}$$

- Challenge: ACL2 is untyped but Z3 is typed.
- Solution: user adds type assertions to antecedent.
  - These are almost always needed anyways.
  - This requirement is not a significant burden.

Challenge: user defined functions



- Challenge:
  - ACL2 supports arbitrary lisp functions.
  - Z3 functions are more like macros (no recursion).
- Solution:
  - Set up translation for a basic set of functions.
  - Expand non-recursive functions.
  - Expand recursive functions to bounded depth.
  - Expansion done on ACL2's representation: can verify correctness.

Other issues:

- Claims can contain non-polynomial terms.
  - Replace offensive subexpression with a variable.
  - User adds constraints about the variable.
  - These constraints are returned as clauses for ACL2 to prove.
- ACL2 may need hints to discharge clauses returned from the clause processor.
  - Solution: nested hints.
  - These hints tell the clause processor what hints to attach to returned clauses.
- These features provides a very flexible back-and-forth between induction proofs in ACL2 and handling the details of the algebra with Z3.

Example - the theorem

$$\forall\ a\ b\ \gamma \in R, m\ n \in Z. If\ 0 < m < n,\ 0 < gamma < 1. \rightarrow$$
$$\gamma^m((a+b)^2 - 2ab) \geq \gamma^n \cdot 2ab$$

```
(defun f-mul-2 (x) (f-mul 2 x))
(defun f-plus (x y) (+ x y))
(defun f-square (x) (f-mul x x))
(defun f-neg (x) (- x))
(defun f-minus (x y) (f-plus x (f-neg y)))
(defun f-expt (x n) (expt x n))
```

Example - code

```
(defthm demonstration
  (implies (and (and (rationalp a)
                     (rationalp b)
                     (rationalp gamma)
                     (integerp m)
                     (integerp n))
                (and (> gamma 0)
                     (< gamma 1)
                     (> m 0)
                     (< m n)))
           (>= (f-mul (expt gamma m)
                      (f-minus (f-square (f-plus a b))
                               (f-mul (f-mul-2 a) b)))
               (f-mul (foo gamma n)
                      (f-mul (f-mul-2 a) b))))
  :hints ...)
```

Example - code

```
:hints
(("Goal"
  :clause-processor
  (my-clause-processor clause
    '( (:expand ((:functions ((f-mul rationalp)
                              (f-mul-2 rationalp)
                              (f-plus rationalp)
                              (f-square rationalp)
                              (f-neg rationalp)
                              (f-minus rationalp)
                              (f-expt rationalp)))
                 (:expansion-level 1))
                (:python-file "demonstration")
                (:let ((expt_gamma_m (expt gamma m) rationalp)
                       (expt_gamma_n (expt gamma n) rationalp)))
                (:hypothesize ((< expt_gamma_n expt_gamma_m)
                               (> expt_gamma_m 0)
                               (> expt_gamma_n 0)))
                (:use ((:type ())
                       (:hypo ())
                       (:main ())))))))))
```

Outline

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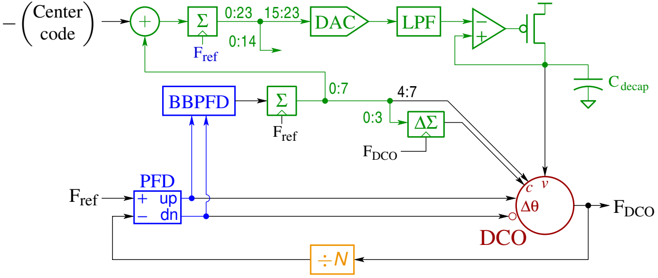
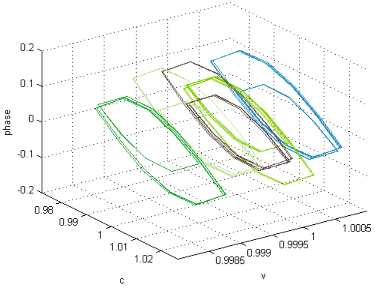
- Contributions
- Integrating SMT with Theorem Proving
- **Verifying global convergence of a Digital Phase-Locked Loop(DPLL) using recurrence**
  - **The state-of-the-art Digital PLL**
  - **Establish recurrence model for the DPLL**
  - **Prove global convergence using Z3 and ACL2**
- Conclusion

A state-of-the-art Digital PLL (from CICC 2010)[CNA10]



- DCO has three control inputs:
  capacitance setting (digital), supply voltage (linear), phase correction (time-difference of digital transitions).
- Uses linear and bang-bang PFD.
- Integrators are digital.
- LPF and decap to improve power-supply rejection.
- It is impractical to verify global convergence using simulation.

Establish the Recurrence Model



A limit cycle is an isolated closed trajectory, for which its neighbouring trajectories are not closed they spiral either towards or away from the limit cycle.

- The recurrence model:

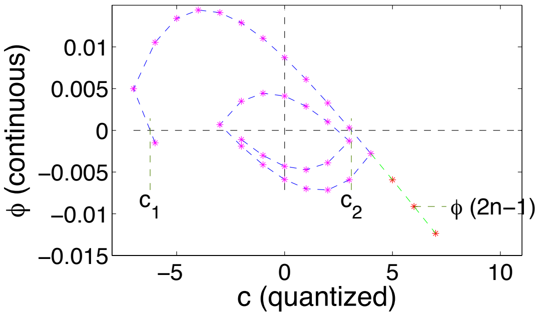
$$\begin{aligned} c(i+1) &= c(i) + g_1 sign(\phi(i)) \\ v(i+1) &= v(i) + g_2(c(i) - c_{code}) \\ \phi(i+1) &= (1-K_t)\phi(i) + 2\pi\left(\frac{f_{dco}(i)}{Nf_{ref}} - 1\right) \end{aligned}$$

where $f_{dco}(i) = f_0 \frac{1+\alpha v(i)}{1+\beta c(i)}$

The proof: the big picture

- Coarse convergence: from any initial condition, ϕ eventually crosses 0 in a state where *c* and *v* are not saturated.
  - Proof sketch:
  - Use Ricatti equation to get a ranking function based on linear model at convergence.
  - Use this ranking function to show coarse convergence using non-linear, global model.
  - Z3 discharges all of the proof obligations.
- Fine convergence: from any crossing of $\phi = 0$ with *c* and *v* away from their saturation conditions (as established above), ϕ will continue to make zero-crossings that each move closer to the intended equilibrium.
  - Proof sketch: see the next few slides.

The proof: fine convergence using induction proof



- Solve the recurrence (verified by ACL2 – rewrite & induction):

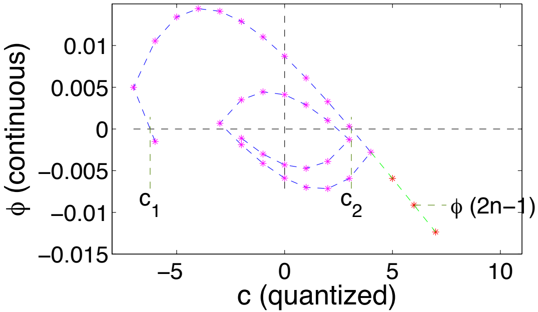
$$\begin{aligned} c(j) &= c_0 + g_1 j \\ \phi(j) &= \gamma^j \phi_0 + 2\pi \sum_{i=0}^{j-1} \gamma^{(j-1-i)} \left(\mu \frac{1+\alpha v}{1+\beta c(i)} - 1 \right) \end{aligned}$$

- We want to prove:

$$\forall m \geq 3,\ \phi(2m-1) < 0$$

A symmetric argument applies to lower half of the space.

The proof



- Exploit the asymmetry between terms with $c < c_{eq}$ and $c > c_{eq}$ where c_{eq} is chosen to set $f_{dco} = f_{ref}$.
  - We pair up points to simplify the formula.
  - Basic idea: the negative terms dominate the positive ones.
- Proving these claims manually involves many pages of messy algebra:
  - Just have Z3 takes care of it.

Conclusion

- We've shown an integration of the Z3 SMT solver into the ACL2 theorem prover with applications for AMS verification.
- Theorem proving is hard! Reachability is easy! Why use a theorem prover?
  - Reachability tools only solve parts of the problem. Human reasoning is needed to conclude that the system works given these partial results.
  - Our formulation lets us work directly on the recurrences rather than on continuizations:
    - Can reason in detail about limit-cycle behaviour.
  - We hope for "re-usable proofs."
    - Proof re-use has been very useful in the hardware verification world.
    - AMS verification seems amenable to the same approach: there aren't that many types of AMS blocks even though there are many implementations.

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